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DECAY OF SOLUTIONS OF DAMPED KIRCHHOFF AND BEAM EQUATIONS

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Abstract. We obtain uniform estimates for solutions of second-order nonlinear nonautonomous differential-operator equation in a Hilbert space with structural damping. It is shown that when the given source term in the equation tends to zero as $t \to \infty$, the corresponding solution of the Cauchy problem for this equation also tends to zero as $t \to \infty$. Exponential decay of solutions for the corresponding autonomous equation is also obtained. Applications to the initial boundary value problems for some nonlinear Kirchhoff type and beam equations are given.

Keywords: Kirchhoff equation, damped beam equation, structural stability, uniform estimates, exponential decay of solutions.

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1. Introduction

We consider the Cauchy problem for differential-operator equation of the form

$$\begin{cases} u_{tt} + \nu A^2 u + \left(\alpha + d\|A^{\frac{1}{2}}u\|^2\right) Au + bA^{\theta}u_t = f(t), \\ u(0) = u_0, u_t(0) = u_1, \end{cases} \tag{1}$$

in a Hilbert space H with inner product $(\cdot,\cdot)$ and the corresponding norm $\|\cdot\|$. Here $A : D(A) \to H$ is a positive definite self-adjoint operator with dense domain $D(A) \subset H$ and compact inverse.

Assume also that $f \in L^2(\mathbb{R}^+; H)$ is a given vector function, $\nu \geq 0,\ \alpha > 0,\ b > 0,\ d > 0;\ \theta \geq 0$ are given parameters. Here and in what follows we are using the notations and inequalities:

- $u_t := \frac{d}{dt}u,\ u_{tt} := \frac{d^2}{dt^2}u.$
- $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n, \cdots$ are the eigenvalues corresponding to the orthonormal system of eigenvectors $w_1, w_2, \cdots, w_n, \cdots$ of the operator A.
- $A^{\theta}u = \sum\limits_{k=1}^{\infty} \lambda_k^{\theta}(u, w_k), \quad \theta \in \mathbb{R}.$
- Poincaré type inequalities:

$$\|A^{\frac{\theta}{2}}u\|^2 \geq \lambda_1^{\theta}\|u\|^2,\ u \in D(A^{\frac{\theta}{2}}), \quad \theta \geq 0; \quad \|A^{\frac{1}{2}}u\|^2 \geq \lambda_1^{1-\theta}\|A^{\frac{\theta}{2}}u\|^2,\ u \in D(A^{\frac{1}{2}}), \quad \theta \in [0,1]. \tag{2}$$

A *strong solution* of the problem (1) is a function $u \in L^{\infty}(0,T; D(A^{1+\frac{\theta}{2}}))$, $u_t \in L^{\infty}(0,T; D(A^{\frac{\theta}{2}}))$ that satisfies the equation (1) in the sense of distributions.
The original nonlinear Kirchhoff equation

$$\partial_t^2(x,t) - \left(\alpha + b\int\limits_0^L [\partial_x u(x,t)]^2 dx\right)\partial_x^2 u(x,t) = 0, \tag{3}$$

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where $\alpha > 0, b > 0$ are given parameters, has been introduced by Kirchhoff [9] to describe the vibration of string of length L with constant cross section. As far as we know the first paper devoted to the mathematical analysis of this equation is the famous paper of S.N. Bernstein [2]. In this paper, it is proved the existence of global in time analytic solution (3) on an interval of real line under the homogeneous Dirichlet's boundary conditions.

Woinowsky-Krieger introduced the equation

$$\partial_t^2(x,t) - \left(\alpha + b\int_0^L [\partial_x u(x,t)]^2 dx\right)\partial_x^2 u(x,t) + \partial_x^4 u(x,t) = 0, \tag{4}$$

to model the transverse motion of an extensible beam.

There are many publications devoted to the problem of global solvability and asymptotic behavior of solutions of the initial-boundary value problems for damped Kirchhoff and beam equations (see, e.g., [3], [5], [7], [8], [10], [11], [13], [15], [16] and references therein). All of these papers are devoted to nonlinear equations with weak, strong or nonlinear damping terms.

We studied the global in time behavior of solutions of the Cauchy problem for second-order nonlinear nonautonomous differential operator equation with structural damping term. Our main results are the estimates (5), (26), (34) and (37) for weak and strong solutions of the problem (1). These estimates imply uniform boundedness of solutions in the case when $\int_0^\infty \|f(t)\|_H$ is bounded. It follows that the solutions tend to zero as $t \to \infty$ when $\|f(t)\| \to 0$ as $t \to \infty$. Moreover the obtained estimates imply exponential decay to zero of solutions of the corresponding autonomous equation (i.e. the equation (1) when $f(t) \equiv 0$.) These results allow us to get decay estimates for solutions of structurally damped Kirchhoff equation, beam equations and some of modifications of these equations.

2. Estimates of solutions

First we prove the following theorem about a weak solution of the problem (1) with $\nu > 0, \ \theta \in [0,2]$, i.e., a function $u \in L^\infty(0,T;D(A)), \ u_t \in L^\infty(0,T;H) \cap L^2(0,T;D(A^{\frac{\theta}{2}}))$ which satisfies the equation (1) in the sense of distributions.

Theorem 2.1. *Suppose that $u_0 \in D(A), \ u_1 \in H$ and $f \in L^2(0,T;H), \ \forall T > 0$. Then, for the weak solution of the problem (1) the following estimate holds true:*

$$\|u_t(t)\|^2 + \nu\|Au(t)\|^2 + \alpha\|A^{\frac{1}{2}}u(t)\|^2 + b\|A^{\frac{\theta}{2}}u(t)\|^2 \le M_0(t), \qquad \forall t \in \mathbb{R}^+, \tag{5}$$

where $M_0(t)$ depends on initial data and the source term f. Moreover $M_0(t) \to 0$ as $t \to \infty$ if $\|f(t)\| \to 0$ as $t \to \infty$.

Proof. Multiplying the equation (1) by $u_t + \varepsilon u$ with some $\varepsilon > 0$, we get

$$\frac{d}{dt}\left[\frac{1}{2}\|u_t\|^2 + \frac{\nu}{2}\|Au\|^2 + \frac{\alpha}{2}\|A^{\frac{1}{2}}u\|^2 + \frac{d}{4}\|A^{\frac{1}{2}}u\|^4 + \varepsilon(u,u_t) + \frac{b\varepsilon}{2}\|A^{\frac{\theta}{2}}u\|^2\right]$$
$$+ \varepsilon\nu\|Au\|^2 + b\|A^{\frac{\theta}{2}}u_t\|^2 - \varepsilon\|u_t\|^2 + \varepsilon\alpha\|A^{\frac{1}{2}}u\|^2 + \varepsilon d\|A^{\frac{1}{2}}u\|^4 = (f(t), u_t + \varepsilon u). \tag{6}$$

By using the Cauchy-Schwarz inequality and the Poincaré type inequality (2) we get

$$\varepsilon\|u_t\|^2 \le \frac{\varepsilon}{\lambda_1^\theta}\|A^{\frac{\theta}{2}}u_t\|^2, \ |(f(t),u_t)| \le \frac{1}{\lambda_1^{\frac{\theta}{2}}}\|A^{\frac{\theta}{2}}u_t\|\|f(t)\| \le \frac{b}{4}\|A^{\frac{\theta}{2}}u_t\|^2 + \frac{1}{b\lambda_1^\theta}\|f(t)\|^2,$$

$$\varepsilon|(f(t),u)| \le \varepsilon\lambda_1^{-\frac{1}{2}}\|f(t)\|\|A^{\frac{1}{2}}u\| \le \frac{\alpha\varepsilon}{4}\|A^{\frac{1}{2}}u\|^2 + \frac{\varepsilon}{\alpha\lambda_1}\|f(t)\|^2.$$

Employing these inequalities we obtain from (6) that

$$\frac{d}{dt}\Phi_\varepsilon(t) + \varepsilon\nu\|Au\|^2 + \left(\frac{3}{4}b - \frac{\varepsilon}{\lambda_1^\theta}\right)\|A^{\frac{\theta}{2}}u_t\|^2 + \frac{3}{4}\alpha\varepsilon\|A^{\frac{1}{2}}u\|^2 + \varepsilon d\|A^{\frac{1}{2}}u\|^4 \le \left(\frac{1}{b\lambda_1^\theta} + \frac{\varepsilon}{\alpha\lambda_1}\right)\|f(t)\|^2, \quad (7)$$

where

$$\Phi_\varepsilon(t) := \frac{1}{2}\|u_t\|^2 + \frac{\nu}{2}\|Au\|^2 + \frac{\alpha}{2}\|A^{\frac{1}{2}}u\|^2 + \frac{d}{4}\|A^{\frac{1}{2}}u\|^4 + \varepsilon(u,u_t) + \frac{b\varepsilon}{2}\|A^{\frac{\theta}{2}}u\|^2. \quad (8)$$

Since $\|A^{\frac{\theta}{2}}u\|^2 \ge \lambda_1^\theta\|u\|^2$, we have $\varepsilon|(u,u_t)| \le \frac{1}{4}\|u_t\|^2 + \varepsilon^2\lambda_1^{-\theta}\|A^{\frac{\theta}{2}}u\|^2$. Therefore, for $\varepsilon \le \frac{b\lambda_1^\theta}{4}$ we obtain the following estimate from below for the function $\Phi_\varepsilon(t)$:

$$\Phi_\varepsilon(t) \ge \frac{1}{4}\|u_t\|^2 + \frac{\nu}{2}\|Au\|^2 + \frac{\alpha}{2}\|A^{\frac{1}{2}}u\|^2 + \frac{d}{4}\|A^{\frac{1}{2}}u\|^4 + \frac{b\varepsilon}{4}\|A^{\frac{\theta}{2}}u\|^2. \quad (9)$$

Let us rewrite (7) in the following form

$$\frac{d}{dt}\Phi_\varepsilon(t) + \delta\Phi_\varepsilon(t) + [H_\varepsilon(t) - \delta\Phi_\varepsilon(t)] \le F_1(t), \quad (10)$$

where $\delta < \varepsilon$ is some positive parameter which will be chosen below, $F_1(t) := \left(\frac{1}{b\lambda_1^\theta} + \frac{\varepsilon}{\alpha\lambda_1}\right)\|f(t)\|^2$ and

$$H_\varepsilon(t) := \varepsilon\nu\|Au\|^2 + \left(\frac{3}{4}b - \frac{\varepsilon}{\lambda_1^\theta}\right)\|A^{\frac{\theta}{2}}u_t\|^2 + \frac{3}{4}\alpha\varepsilon\|A^{\frac{1}{2}}u\|^2 + \varepsilon d\|A^{\frac{1}{2}}u\|^4.$$

Let us show that $\varepsilon > 0,\ \delta > 0$ can be chosen so that $H_\varepsilon(t) - \delta\Phi_\varepsilon(t) \ge 0$. In fact employing, the inequalities

$$\varepsilon\|u_t\|^2 \le \varepsilon\lambda_1^{-\theta}\|A^{\frac{\theta}{2}}u_t\|^2, \quad (11)$$

$$\delta\varepsilon|(u,u_t)| \le \frac{1}{2}\delta\varepsilon\lambda_1^{-\theta}\|A^{\frac{\theta}{2}}u_t\|^2 + \frac{1}{2}\delta\varepsilon\lambda_1^{-1}\|A^{\frac{1}{2}}u\|^2 \quad (12)$$

we obtain

$$H_\varepsilon(t) - \delta\Phi_\varepsilon(t) \ge \left(\varepsilon\nu - \frac{\delta\nu}{2}\right)\|Au\|^2 + \left(\frac{3}{4}b - \frac{\varepsilon}{\lambda_1^\theta} - \frac{1}{2}\delta\varepsilon\lambda_1^{-\theta} - \frac{\delta}{2\lambda_1^\theta}\right)\|A^{\frac{\theta}{2}}u_t\|^2 + \left(\frac{3}{4}\alpha\varepsilon - \frac{1}{2}\delta\alpha - \frac{1}{2}\delta\varepsilon\lambda_1^{-1} - \frac{\delta b\varepsilon}{2\lambda_1^{\theta-1}}\right)\|A^{\frac{1}{2}}u\|^2 + (\varepsilon - \frac{\delta d}{4})\|A^{\frac{1}{2}}u\|^4. \quad (13)$$

First we choose $\varepsilon = \frac{1}{4}b\lambda_1^\theta$. Then, we can choose $\delta < \varepsilon$ small enough and see that $H_\varepsilon(t) - \delta\Phi_\varepsilon(t) \ge \frac{b}{4}\|A^{\frac{1}{2}}u_t\|^2$. Thus (6) implies the inequality

$$\frac{d}{dt}\Phi_\varepsilon(t) + \delta\Phi_\varepsilon(t) + \frac{b}{4}\|A^{\frac{\theta}{2}}u_t\|^2 \le F_1(t). \quad (14)$$

Hence

$$\Phi_\varepsilon(t) \le \Phi_\varepsilon(0)e^{-\delta t} + e^{-\delta t}\int_0^t e^{\delta\tau}\|F_1(\tau)\|^2 d\tau. \quad (15)$$

From this inequality and (9) we deduce that

$$\|u_t\|^2 + \nu\|Au\|^2 + \alpha\|A^{\frac{1}{2}}u\|^2 + b\|A^{\frac{\theta}{2}}u\|^2 \le Q_0 e^{-\delta t} + Q_1\|f\|^2_{L^2(0,t;H)} := M_0(t), \quad \forall t > 0, \quad (16)$$

where $Q_0 > 0$ is a parameter depending on initial data, Q_1 depends on the parameters of the equation.
On the other hand (15) implies that if $\|f(t)\| \to 0$ as $t \to \infty$ then

$$\mathcal{E}_1(t) := \|u_t\|^2 + \nu\|Au\|^2 + \alpha\|A^{\frac{1}{2}}u\|^2 + b\|A^{\frac{\theta}{2}}u\|^2 \to 0, \quad \text{as} \quad t \to \infty. \tag{17}$$

□

Let us consider the problem (1) with $\nu = 0$ and $\theta \in (0, 1]$, i.e. the problem

$$\begin{cases} u_{tt} + \left(\alpha + d\|A^{\frac{1}{2}}u\|^2\right) Au + bA^\theta u_t = f(t), \\ u(0) = u_0, u_t(0) = u_1. \end{cases} \tag{18}$$

The equality (6) in this case takes the form

$$\frac{d}{dt}\left[\frac{1}{2}\|u_t\|^2 + \frac{\alpha}{2}\|A^{\frac{1}{2}}u\|^2 + \frac{d}{4}\|A^{\frac{1}{2}}u\|^4 + \varepsilon(u, u_t) + \frac{b\varepsilon}{2}\|A^{\frac{\theta}{2}}u\|^2\right]$$
$$+ b\|A^{\frac{\theta}{2}}u_t\|^2 - \varepsilon\|u_t\|^2 + \varepsilon\alpha\|A^{\frac{1}{2}}u\|^2 + \varepsilon d\|A^{\frac{1}{2}}u\|^4 = (f(t), u_t + \varepsilon u). \tag{19}$$

Now, we use the inequalities

$$|(\varepsilon u, f)| \le \varepsilon\lambda_1^{-\frac{1}{2}}\|A^{\frac{1}{2}}u\|\|f(t)\| \le \frac{1}{2}\varepsilon\alpha\|A^{\frac{1}{2}}u\|^2 + \frac{\varepsilon}{2\alpha\lambda_1}\|f(t)\|^2$$

$$\varepsilon\|u_t\|^2 \le \varepsilon\lambda_1^{-\theta}\|A^{\frac{\theta}{2}}u_t\|^2, \ |(f(t), u_t)| \le \lambda_1^{-\frac{\theta}{2}}\|A^{\frac{\theta}{2}}u_t\|\|f(t)\| \le \varepsilon\lambda_1^{-\theta}\|A^{\frac{\theta}{2}}u_t\|^2 + \frac{1}{4\varepsilon}\|f(t)\|^2$$

and infer from (19) that

$$\frac{d}{dt}J_\varepsilon(t) + (b - 2\varepsilon\lambda_1^{-\theta})\|A^{\frac{\theta}{2}}u_t\|^2 + \frac{1}{2}\varepsilon\alpha\|A^{\frac{1}{2}}u\|^2 + \varepsilon d\|A^{\frac{1}{2}}u\|^4 \le \left(\frac{\varepsilon}{2\alpha\lambda_1} + \frac{1}{4\varepsilon}\right)\|f(t)\|^2, \tag{20}$$

where

$$J_\varepsilon(t) := \frac{1}{2}\|u_t\|^2 + \frac{\alpha}{2}\|A^{\frac{1}{2}}u\|^2 + \frac{d}{4}\|A^{\frac{1}{2}}u\|^4 + \varepsilon(u, u_t) + \frac{b\varepsilon}{2}\|A^{\frac{\theta}{2}}u\|^2.$$

By choosing $\varepsilon \le \varepsilon_0 := \frac{1}{4}\lambda_1^\theta b$ we obtain from (20) the inequality

$$\frac{d}{dt}J_\varepsilon(t) + \frac{b}{2}\|A^{\frac{\theta}{2}}u_t\|^2 + \frac{1}{2}\varepsilon_0\alpha\|A^{\frac{1}{2}}u\|^2 + \varepsilon_0 d\|A^{\frac{1}{2}}u\|^4 \le D_0\|f(t)\|^2, \tag{21}$$

with $D_0 := \frac{\varepsilon_0}{2\alpha\lambda_1} + \frac{1}{4\varepsilon_0}$. Similar to how it was done when deriving (14) we can choose $\delta_1 > 0$ small enough such that

$$\frac{d}{dt}J_\varepsilon(t) + \delta_1 J_\varepsilon(t) \le D_0\|f(t)\|^2.$$

Integrating the last inequality

$$\|u_t(t)\|^2 + \alpha\|A^{\frac{1}{2}}u(t)\|^2 \le Q_0 e^{-\delta t} + Q_1\|f\|^2_{L^2(0,t;H)} := M_{01}(t), \quad \forall t > 0. \tag{22}$$

Now, we will obtain estimate of a strong solution to the problem (18) with $\theta \in [\frac{1}{2}, 1]$, i.e. a function $u \in L^\infty(\mathbb{R}^+; D(A^\theta)), \quad u_t \in L^\infty(\mathbb{R}^+; H)$, that satisfies the equation (18) in the sense of distributions.
Assume that

$$u_0 \in D(A^\theta), \ u_1 \in H, \ f \in L^2(\mathbb{R}^+; H).$$

Multiplication of (18) by $A^\theta u$ gives

$$\frac{d}{dt}\mathcal{E}_2(t) - \|A^{\frac{\theta}{2}}u_t\|^2 + \alpha\|A^{\frac{1}{2}+\frac{\theta}{2}}u\|^2 + d\|A^{\frac{1}{2}}u\|^2\|A^{\frac{\theta}{2}+\frac{1}{2}}u\|^2 = (f(t), A^\theta u), \tag{23}$$

where
$$\mathcal{E}_2(t) := \frac{b}{2}\|A^\theta u\|^2 + (u_t, A^\theta u).$$
Next, we multiply (23) by $\eta > 0$ and add to (21):
$$\frac{d}{dt}\left[J_{\varepsilon_0}(t) + \eta\mathcal{E}_2(t)\right] + \left(\frac{b}{2} - \eta\right)\|A^{\frac{\theta}{2}}u_t\|^2 + \eta\alpha\|A^{\frac{1}{2}+\frac{\theta}{2}}u\|^2 + \frac{1}{2}\varepsilon_0\alpha\|A^{\frac{1}{2}}u\|^2 + \varepsilon_0 d\|A^{\frac{1}{2}}u\|^4$$
$$\le D_0\|f(t)\|^2 + \eta|(f(t), A^\theta u)|. \quad (24)$$
Since
$$\eta|(f(t), A^\theta u)| \le \eta\|f(t)\|\|A^\theta u)\| \le \eta\lambda_1^{(\theta-1)/2}\|A^{\frac{1}{2}+\frac{\theta}{2}}u\|\|f(t)\|$$
$$\le \frac{1}{2}\eta\alpha\|A^{\frac{1}{2}+\frac{\theta}{2}}u\|^2 + \frac{\eta}{2\alpha}\lambda_1^{\theta-1}\|f(t)\|^2,$$
we use the last inequality in (24) and choose $\eta \in (0, \frac{b}{4})$ we obtain
$$\frac{d}{dt}\left[J_{\varepsilon_0}(t) + \eta\mathcal{E}_2(t)\right] + \frac{b}{4}\|A^{\frac{\theta}{2}}u_t\|^2 + \frac{1}{2}\eta\alpha\|A^{\frac{1}{2}+\frac{\theta}{2}}u\|^2 + \frac{1}{2}\varepsilon_0\alpha\|A^{\frac{1}{2}}u\|^2 + \varepsilon_0 d\|A^{\frac{1}{2}}u\|^4$$
$$\le (D_0 + \frac{\eta}{2\alpha}\lambda_1^{\theta-1})\|f(t)\|^2. \quad (25)$$
Similar to (13) we can show that for some $\delta_2 > 0$,
$$\frac{b}{4}\|A^{\frac{\theta}{2}}u_t\|^2 + \frac{1}{2}\eta\alpha\|A^{\frac{1}{2}+\frac{\theta}{2}}u\|^2 + \frac{1}{2}\varepsilon_0\alpha\|A^{\frac{1}{2}}u\|^2 + \varepsilon_0 d\|A^{\frac{1}{2}}u\|^4 - \delta_2\left[J_{\varepsilon_0}(t) + \eta\mathcal{E}_2(t)\right] \ge 0, \ \forall t \ge 0$$
and deduce form (25) that
$$\frac{d}{dt}\left[J_{\varepsilon_0}(t) + \eta\mathcal{E}_2(t)\right] + \delta_2\left[J_{\varepsilon_0}(t) + \eta\mathcal{E}_2(t)\right] \le (D_0 + \frac{\eta}{2\alpha}\lambda_1^{\theta-1})\|f(t)\|^2,$$
which implies that
$$J_{\varepsilon_0}(t) + \eta\mathcal{E}_2(t) \le G(t)$$
where $G(t) \to 0$ as $t \to 0$ if $\|f(t)\| \to 0$ as $t \to \infty$. Remembering that $\|u_t(t)\|^2 \le M_{01}(t)$ (see (22)) and taking into account the inequality
$$\mathcal{E}_2(t) \ge \frac{b}{4}\|A^\theta u\|^2 - \frac{1}{b}\|u_t\|^2,$$
we obtain the desired result:

Theorem 2.2. *If $u(t)$ is a strong solution of the problem* (1) *with $\nu = 0$, $\theta \in [\frac{1}{2}, 1]$, $u_0 \in D(A^\theta)$, $u_1 \in H$ and $f \in L^2(0, T; H)$, then the strong solution of the problem satisfies the estimate*
$$\|A^\theta u(t)\|^2 + \|u_t(t)\|^2 \le R(t), \quad \forall \in \mathbb{R}^+, \quad (26)$$
where $R(t) \to 0$ as $t \to \infty$ when $\|f(t)\| \to 0$ as $t \to \infty$.
Moreover if $f(t) \equiv 0$, then $\|A^\theta u_t(t)\|^2 + \|u(t)\|^2$ tends to zero as $t \to \infty$ with an exponential rate.

Decay estimates for the problem (1) **with** $\nu > \mathbf{0}$: Assume that u is a strong solution of the problem (1) with $\nu > 0$, $\theta \in [1, 2]$ that is the function $u \in L^\infty(\mathbb{R}^+; D(A^{1+\frac{\theta}{2}}))$ with $u_t \in L^\infty(\mathbb{R}^+; D(A^{\frac{\theta}{2}}))$ which satisfies the equation (1) in the sense of distributions and suppose that $u_0 \in D(A^{1+\frac{\theta}{2}})$, $u_1 \in D(A^\theta)$, $f \in L^2(\mathbb{R}^+; H)$. First we multiply the equation (1) by $A^\theta u$:
$$\frac{d}{dt}\left[(u_t, A^\theta u) + \frac{b}{2}\|A^\theta u\|^2\right] - \|A^{\frac{\theta}{2}}u_t\|^2 + \nu\|A^{1+\frac{\theta}{2}}u\|^2 + \alpha\|A^{\frac{1}{2}+\frac{\theta}{2}}u\|^2 \le (f(t), A^\theta u).$$

Addition of the last inequality multiplied by $\eta > 0$ with (14) gives us the inequality

$$\frac{d}{dt}U_\eta(t) + \eta\nu\|A^{1+\frac{\theta}{2}}u\|^2 + \eta\alpha\|A^{\frac{1}{2}+\frac{\theta}{2}}u\|^2 + \delta\Phi_\varepsilon(t) + (\frac{b}{4} - \eta)\|A^{\frac{\theta}{2}}u_t\|^2 \leq F_1(t) + \eta(f(t), A^\theta u), \quad (27)$$

where

$$U_\eta := \Phi_\varepsilon(t) + \left[\eta(u_t, A^\theta u) + \frac{\eta b}{2}\|A^\theta u\|^2\right].$$

Utilizing the inequality

$$\eta|(f(t), A^\theta u)| \leq \eta\lambda_1^{\theta-2}\|f(t)\|\|A^{1+\frac{\theta}{2}}u\| \leq \frac{\eta\nu}{2}\|A^{1+\frac{\theta}{2}}u\|^2 + \frac{\eta}{2\nu\lambda_1^{4-2\theta}}\|f(t)\|^2$$

and choosing $\eta \leq \frac{b}{8}$ we can rewrite (27) in the following form

$$\frac{d}{dt}U_\eta(t) + \kappa U_\eta(t) + \mathcal{H}(t) - \kappa U_\eta(t) \leq \mathcal{F}(t), \quad (28)$$

where $\kappa > 0$,

$$\mathcal{H}(t) := \frac{\eta\nu}{2}\|A^{1+\frac{\theta}{2}}u\|^2 + \eta\alpha\|A^{\frac{1}{2}+\frac{\theta}{2}}u\|^2 + \delta\Phi_\varepsilon(t) + (\frac{b}{4} - \eta)\|A^{\frac{\theta}{2}}u_t\|^2,$$

and $\mathcal{F}(t) := F_1(t) + \frac{\eta}{2\nu\lambda_1^{4-2\theta}}\|f(t)\|^2$. Now, we can choose $\kappa > 0$ small enough (depending on the parameters of the equation) such that

$$\mathcal{H}(t) - \kappa U_\eta(t) \geq \kappa_1\left[\|A^{1+\frac{\theta}{2}}u\|^2 + \|A^{\frac{\theta}{2}}u_t\|^2\right],$$

with some $\kappa_1 > 0$ and hence

$$\frac{d}{dt}U_\eta(t) + \kappa U_\eta(t) + \kappa_1\left[\|A^{1+\frac{\theta}{2}}u\|^2 + \|A^{\frac{\theta}{2}}u_t\|^2\right] \leq \mathcal{F}(t). \quad (29)$$

Integrating the last inequality we get the estimate

$$\|u_t\|^2 + \|Au\|^2 + \|A^\theta u\|^2 \leq M_1^*(t), \quad \forall t \in \mathbb{R}^+. \quad (30)$$

Now we multiply the equation (1) by $A^\theta u_t$ and obtain that

$$\frac{1}{2}\frac{d}{dt}\left[\|A^{\frac{\theta}{2}}u_t\|^2 + \nu\|A^{1+\frac{\theta}{2}}u\|^2 + \alpha\|A^{\frac{1}{2}+\frac{\theta}{2}}u\|^2\right]$$
$$+ d\|A^{\frac{1}{2}}u\|^2(Au, A^\theta u_t) + b\|A^\theta u_t\|^2 \leq (f(t), A^\theta u_t) \leq \frac{b}{4}\|A^\theta u_t\|^2 + \frac{1}{b}\|f(t)\|^2. \quad (31)$$

On the other hand

$$\|A^{\frac{1}{2}}u\|^2|(Au, A^\theta u_t)| \leq \frac{b}{4}\|A^\theta u_t\|^2 + \frac{1}{b}\|A^{\frac{1}{2}}u\|^4\|Au\|^2$$
$$\leq \frac{b}{4}\|A^\theta u_t\|^2 + \frac{1}{b}\lambda_1^{2-2\theta}\|A^{\frac{1}{2}}u\|^4\|A^\theta u\|^2 \leq \frac{b}{4}\|A^\theta u_t\|^2 + \frac{1}{b}\lambda_1^{2-2\theta}M_0(t)^2R(t).$$

Thus (31) implies:

$$\frac{d}{dt}\left[\|A^{\frac{\theta}{2}}u_t\|^2 + \nu\|A^{1+\frac{\theta}{2}}u\|^2 + \alpha\|A^{\frac{1}{2}+\frac{\theta}{2}}u\|^2\right] + \frac{b}{2}\|A^\theta u_t\|^2 \leq \frac{2}{b}\lambda_1^{2-2\theta}M_0(t)^2R(t). \quad (32)$$

Adding the last inequality and (29) yields

$$\frac{d}{dt}V_\gamma(t) + \kappa U_\eta(t) + \kappa_1\left[\|A^{1+\frac{\theta}{2}}u\|^2 + \|A^{\frac{\theta}{2}}u_t\|^2\right] \leq \mathcal{G}(t), \quad (33)$$

where

$$V_\gamma(t) := \|A^{\frac{\theta}{2}}u_t\|^2 + \nu\|A^{1+\frac{\theta}{2}}u\|^2 + \alpha\|A^{\frac{1}{2}+\frac{\theta}{2}}u\|^2 + U_\eta(t)$$

and $\mathcal{G}(t) := \mathcal{F}(t) + \frac{2}{b}\lambda_1^{2-2\theta}M_0(t)^2R(t)$. The last inequality implies existence of $\gamma > 0$ such that

$$\frac{d}{dt}V_\gamma(t) + \gamma V_\gamma(t) \le \mathcal{G}(t).$$

Integrating this inequality we obtain the estimate

$$\|A^{\frac{\theta}{2}}u_t\|^2 + \nu\|A^{1+\frac{\theta}{2}}u\|^2 + \alpha\|A^{\frac{1}{2}+\frac{\theta}{2}}u\|^2 \le Q_0V_\gamma(0)e^{-\gamma t} + Q_1\int_0^t \mathcal{G}(\tau)d\tau.$$

So we proved the following Theorem:

Theorem 2.3. *If $u_0 \in D(A^{1+\frac{\theta}{2}}),\ u_1 \in D(A^\theta),\ f \in L^2(\mathbb{R}^+; H)$ and u is a strong solution of the problem (1) with $\nu > 0,\ \theta \in [1, 2]$ Then, the following estimate holds true:*

$$\|A^{\frac{\theta}{2}}u_t(t)\|^2 + \nu\|A^{1+\frac{\theta}{2}}u(t)\|^2 \le L_0(t), \qquad \forall t \in \mathbb{R}^+, \tag{34}$$

where $L_0(t)$ depends on initial data and the source term f. Moreover $L_0(t) \to 0$ as $t \to \infty$ if $\|f(t)\| \to 0$ as $t \to \infty$.

Analog of the Theorem 2.1, i.e., the following corollary holds also for solutions of the problem

$$\begin{cases} u_{tt} + \nu A^2u + \left(\alpha + d\|A^{\frac{1}{2}}u\|^2\right)Au + bA^\theta u_t + (Au, u_t)Au = f(t), \\ u(0) = u_0, \quad u_t(0) = u_1. \end{cases} \tag{35}$$

Theorem 2.4. *If $\theta \in [0, 2],\ u_1 \in H,\ u_0 \in D(A)$ and $f \in L^2(0, T; H)$, then the weak solution of the problem (35) satisfies the estimate*

$$\mathbb{I}_1(t) := \|u_t(t)\|^2 + \|Au(t)\|^2 \le M_2(t), \quad \forall \in \mathbb{R}^+. \tag{36}$$

If $\theta \in [1, 2],\ u_1 \in D(A^\theta),\ u_0 \in D(A^{1+\frac{\theta}{2}})$ and $f \in L^2(0, T; H)$, then

$$\mathbb{I}_2(t) := \|A^{\frac{\theta}{2}}u_t(t)\|^2 + \|A^{1+\frac{\theta}{2}}u(t)\|^2 \le M_3(t), \quad \forall \in \mathbb{R}^+. \tag{37}$$

where $M_j(t) \to 0$ $(j = 2, 3)$ as $t \to \infty$ and $\mathbb{I}_1(t),\ \mathbb{I}_2(t)$ tend to zero with an exponential rate, if $f(t) \equiv 0$.

In fact multiplication of (35) by $u_t + \varepsilon u$ gives

$$\begin{aligned} &\frac{d}{dt}\left[\frac{1}{2}\|u_t\|^2 + \frac{\nu}{2}\|Au\|^2 + \frac{\alpha}{2}\|A^{\frac{1}{2}}u\|^2 + \frac{d+\varepsilon}{4}\|A^{\frac{1}{2}}u\|^4 + \varepsilon(u, u_t) + \frac{b\varepsilon}{2}\|A^{\frac{\theta}{2}}u\|^2\right] \\ &\qquad + (Au, u_t)^2 + b\|A^{\frac{\theta}{2}}u_t\|^2 - \varepsilon\|u_t\|^2 + \varepsilon\nu\|Au\|^2 + \varepsilon\alpha\|A^{\frac{1}{2}}u\|^2 + \varepsilon d\|A^{\frac{1}{2}}u\|^4 \\ &\qquad\qquad = (f(t), u_t + \varepsilon u) \le \varepsilon\|u_t\|^2 + \varepsilon\|u\|^2 + \left(\frac{1}{4\varepsilon} + \frac{\varepsilon}{4}\right)\|f(t)\|^2. \end{aligned} \tag{38}$$

Employing the inequality (2) and choosing $\varepsilon < \frac{1}{2}\{b\lambda_1^\theta, \nu\lambda_1^2\}$ we obtain from (38) that

$$\frac{d}{dt}[\Phi_\varepsilon(t) + \frac{\varepsilon}{4}\|A^{\frac{1}{2}}\|^4] + \frac{\nu}{2}\|Au\|^2 + (Au, u_t)^2 + \frac{b}{2}\|A^{\frac{\theta}{2}}u_t\|^2 + \varepsilon\alpha\|A^{\frac{1}{2}}u\|^2 \le \left(\frac{1}{4\varepsilon} + \frac{\varepsilon}{4}\right)\|f(t)\|^2, \tag{39}$$

where $\Phi_\varepsilon(t)$ is defined in (8). Having the last estimate we can deduce the estimate (36) similar to the proof of Theorem 2.1. If $\theta \in [1, 2]$ we multiply (35) by $A^\theta u_t + \sigma A^\theta u$:

$$\begin{aligned} &\frac{d}{dt}H_\sigma(t) + 2b\|A^\theta u_t\|^2 + 2d\|A^{\frac{1}{2}}u\|^2(Au, A^\theta u_t) + 2\sigma\nu\|A^{1+\frac{\theta}{2}}u\|^2 + 2\sigma\alpha\|A^{\frac{1}{2}+\frac{\theta}{2}}u\|^2 \\ &\qquad + 2\sigma d\|A^{\frac{1}{2}}u\|^2(Au, A^\theta u) - 2\sigma\|A^{\frac{\theta}{2}}u_t\|^2 = 2(f(t), A^\theta u_t + \sigma A^\theta u), \end{aligned} \tag{40}$$

where

$$H_\sigma(t) := \|A^{\frac{\theta}{2}}u_t\|^2 + \nu\|A^{1+\frac{\theta}{2}}u\|^2 + \alpha\|A^{\frac{1}{2}+\frac{\theta}{2}}u\|^2 + 2\sigma(u_t, A^\theta u) + \sigma b\|A^\theta u\|^2.$$

Utilizing the Poincaré type inequalities (2) and the estimate (36) we get:

$$2d\|A^{\frac{1}{2}}u\|^2|(Au, A^\theta u_t)| \le \varepsilon_2\|A^\theta u_t\|^2 + \frac{d^2}{\varepsilon_2}\|A^{\frac{1}{2}}u\|^4\|Au\|^2 \le \varepsilon_2\|A^\theta u_t\|^2 + \frac{d^2}{\lambda_1^2\varepsilon_2}[M_2(t)]^6,$$

$$\begin{aligned}2\sigma d\|A^{\frac{1}{2}}u\|^2|(Au, A^\theta u)| &\le 2\sigma d\lambda_1^{\theta-2}\|A^{\frac{1}{2}}u\|^2\|Au\|\|A^{\frac{1}{2}+\frac{\theta}{2}}u\| \\ &\le \varepsilon_3\|A^{\frac{1}{2}+\frac{\theta}{2}}u\|^2 + \frac{1}{\varepsilon_3}[\sigma d\lambda_1^{\theta-3}]^2[M_2(t)]^6,\end{aligned}$$

$$2|(f(t), A^\theta u_t + \sigma A^\theta u)| \le \varepsilon_2\|A^{\frac{\theta}{2}}u_t\|^2 + \varepsilon_3\|A^{1+\frac{\theta}{2}}u\|^2 + \left(\frac{1}{\varepsilon_2} + \frac{\sigma^2}{\lambda_1^{4-2\theta}}\right)\|f(t)\|^2.$$

By using last three inequalities and the inequality $2\sigma\|A^{\frac{\theta}{2}}u_t\|^2 \le 2\sigma\lambda_1^{-\theta}\|A^\theta u_t\|^2$ in (40) we obtain

$$\frac{d}{dt}H_\sigma(t) + (2b - \frac{2\sigma}{\lambda_1^\theta} - \varepsilon_2)\|A^\theta u_t\|^2 + (2\sigma\nu - 2\varepsilon_3)\|A^{1+\frac{\theta}{2}}u\|^2 + 2\sigma\alpha\|A^{\frac{1}{2}+\frac{\theta}{2}}u\|^2 \le \mathcal{F}_1(t), \qquad (41)$$

where

$$\mathcal{F}_1(t) := \left[\frac{d^2}{\lambda_1^2\varepsilon_2} + \frac{1}{\varepsilon_3}[\sigma d\lambda_1^{\theta-3}]^2\right][M_2(t)]^6 + \left(\frac{1}{\varepsilon_2} + \frac{\sigma^2}{\lambda_1^{4-2\theta}}\right)\|f(t)\|^2.$$

Finally we choose $\sigma = \frac{1}{2}b\lambda_1^\theta$ and $\varepsilon_2 > 0,\ \varepsilon_3 > 0$ small enough and deduce from (41) the desired estimate (37).

Remark 2.1. *For the results on existence and uniqueness of solutions of the Kirchhoff equation and the Beam equation we refer to* [4], [6], [12], [14] *and references therein. In* [4] *the author studied the problem of long time dynamics of the autonomous Kirchhoff equation with structural damping term, in* [6] *the author proved global solvability of initial boundary value problem for the extensible beam equation with strong damping term.*

3. Applications

The Kirchhoff Equation with structural damping: Here we consider the problem:

$$\begin{cases}\partial_t^2 u - \alpha\Delta u - d\|\nabla u\|^2_{L^2(\Omega)}\Delta u + (-\Delta)^\theta\partial_t u = f(x,t), x \in \Omega,\ t > 0,\\ u(x,0) = u_0(x),\ \partial_t u(x,0) = u_1, \quad x \in \partial\Omega,\\ u(x,t) = 0,\ x \in \partial\Omega, t > 0,\end{cases} \qquad (42)$$

where $d > 0,\ \beta > 0,\ \alpha > 0,\ \theta \in [\frac{1}{2}, 1]$ are given numbers, $\Omega \subset \mathbb{R}^n$ is a bounded domain with sufficiently smooth boundary $\partial\Omega$ and $f \in L^2(0,T; L^2(\Omega)),\ \ \forall T > 0$ is a given source term. This equation can be written in the form (1), where $\nu = 0$ and A is the Laplace operator $-\Delta$ under the homogeneous Dirichlet's boundary condition. Then according to Theorem 2.2, the following estimate holds ture

$$\|\partial_t u(t)\|^2_{L^2(\Omega)} + \alpha\|u(t)\|^2_{H^{2\theta}(\Omega)} \le R(t), \qquad (43)$$

where $\|\partial_t u(t)\|^2_{H^\theta(\Omega)} + \alpha\|u(t)\|^2_{H^{1+\theta}(\Omega)}$ tends to zero with an exponential rate, if $\|f(t)\|_{L^2(\Omega)} \to 0$ as $t \to \infty$.

Structurally damped beam equation: Here we consider the problem:

$$\begin{cases} \partial_t^2 u + \nu \partial_x^4 u - \left(\alpha + d \int\limits_0^L u_x^2(x,t)dx \right) \partial_x^2 u + b(-\partial_x^2)^\theta \partial_t u = f(x,t), x \in (0,L), t > 0, \\ u(x,0) = u_0(x),\ \partial_t u(x,0) = u_1, \quad x \in (0,L), \\ u(0,t) = u(L,t) = \partial_x^2 u(0,t) = \partial_x^2 u(0,t),\ t > 0, \end{cases} \tag{44}$$

where

$$d > 0,\ b > 0,\ \nu > 0,\ \alpha > 0,\ \theta \in [0,2] \text{ are given numbers, } f \in L^2(0,T;L^2(0,L)),\ \forall T > 0, \tag{45}$$

is a given source term. This equation can be written in the form (1), where A is the Sturm-Liuoville operator $-\frac{d^2}{dx^2}$ under the homogeneous Dirichlet's boundary conditions and the domain of definition $H^2(0,L) \cap H_0^1(0,L)$. According to the Theorem 2.3 we have

$$\mathcal{E}_3(t) := \|\partial_t u(t)\|^2_{\mathcal{H}^\theta(0,L} + \nu \|u(t)\|^2_{H^{2+\theta}(0,L)} \leq M_3(t),$$

where $M_3(t) \to 0,\ t \to \infty$, if $\|f(t)\|_{L^2(\Omega)} \to 0$ as $t \to \infty$. If $f(t) \equiv 0$, then $\mathcal{E}_3(t)$ tends to zero with an exponential rate as $t \to \infty$.

Damped extensible beam equation: Finally, we consider the initial boundary value problem for the equation describing the transverse motion of an extensible beam with structural and external damping terms: (see [1])

$$\begin{cases} \partial_t^2 u + \nu \partial_x^4 u - \left(\alpha + d \int\limits_0^L [\partial_x u]^2 dx + \int\limits_0^L \partial_t \partial_x u \partial_x u \right) \partial_x^2 u + b(-\partial_x^2)^\theta \partial_t u = f(x,t), \\ u(x,0) = u_0(x),\ \partial_t u(x,0) = u_1, \\ u(0,t) = u(L,t) = \partial_x^2 u(0,t) = \partial_x^2 u(0,t),\ t > 0. \end{cases}$$

Here the parameters α, ν, b, d and the function f satisfy (45). According to the Theorem 2.4 the following estimate for the weak solution of this problem holds true

$$\mathcal{E}_4(t) := \|\partial_t u(t)\|^2_{H^\theta(0;l)} + \nu \|u(t)\|^2_{H^2(0,L)} \leq R_1(t),$$

where $R_1(t) \to 0,\ t \to \infty$, if $\|f(t)\|_{L^2(\Omega)} \to 0$ as $t \to \infty$, if $f(t) \equiv 0$, then $\mathcal{E}_4(t)$ exponentially tends to zero as $t \to \infty$.

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